Excess Returns and Beta: Deriving the Security Market Line

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I. We showed that market forces combined with a search by investors for efficient portfolios would produce the following relationships for each security (*i, j* =1, ..., *n*) in a portfolio:

(1) $$\frac{R_i - R_f}{\partial s_m / \partial X_i} = \frac{R_m - R_f}{s_m}$$

This expression can be rewritten as

(2) $$R_i = R_f + \left(\frac{R_m - R_f}{s_m} \right) \partial s_m / \partial X_i$$

II. Expression (2) says that the equilibrium expected return (R_i) on security *i* should equal the risk-free rate (R_f) plus the market price of risk $(R_m - R_f)/s_m$ times $\partial s_m / \partial X_i$. What is $\partial s_m / \partial X_i$? It is the increase in risk (s_m) associated with a small increase in asset *i*, in other words, it is the risk contribution of security *i* to portfolio risk, s_m. We can derive $\partial s_m / \partial X_i$ by taking the derivative of the expression for the variance of a portfolio with respect to X_i. The result, as shown in Garbade (p. 175, fn 12) is:

(3) $$\frac{\partial s_m}{\partial X_i} = \frac{1}{s_m} \sum_{j=1}^{n} X_j Cov(R_i, R_j)$$

This expression states that the risk contribution of a security to the portfolio depends on the covariance of asset *i* with each and every other asset. This makes considerable sense because we know that only systematic risk matters in a portfolio and systematic risk is measured by covariance.

III. The problem with expression (3) above is that it is not operational; there are too many covariances needed to measure the risk of security *i*. We can simplify the measurement problem by recalling the regression equation (security characteristic line) relating the return on an individual security, R_i, to the return on an index, R_m, consisting of all other securities in the market. In particular, we have:

(4) $$R_i = a_i + b_i R_m + e_i$$

Note that R_m is defined as the weighted average return on all securities in the market:

(5) $$R_m = \sum_{j=1}^{n} X_j R_j .$$

From statistics we know that the definition of the regression coefficient in expression (4) is given by:

(6) $$b_i = \frac{Cov(R_i, R_m)}{s_m^2}$$

This means that we can take the expression for R_m in (5) and substitute it into expression (6) to get the following:

(7) $$b_i = \frac{Cov\left(R_i, \sum_{i=1}^{n} X_j R_j\right)}{s_m^2}$$

Expression (7) can be simplified as follows:

(8) $$b_i = \frac{\sum_{j=1}^{n} X_j Cov(R_{i,} R_j)}{s_m^2}$$

(9) $$b_i = \frac{1}{s_m} \cdot \left[\frac{1}{s_m} \cdot \sum_{j=1}^{n} X_j Cov(R_i, R_j) \right]$$

Notice that the expression inside the brackets in (9) is identical to the expression for $\partial s_m / \partial X_i$ in (3). Thus, we can use expression (3) to rewrite (9), as follows:

(10) $$b_i = \frac{1}{s_m} \cdot \partial s_m / \partial X_i$$

IV. Finally, we can produce the result we want. We can use expression (10) to simplify expression (2), which is where we started, and make it operational. From (10) we have

(11) $$\partial s_m / \partial X_i = b_i s_m$$

We can then substitute (11) into (2), which produces

(12) $$R_i = R_f + \left(\frac{R_m - R_f}{s_m} \right) \cdot b_i s_m$$

Which gives us the security market line

(13) $$R_i = R_f + (R_m - R_f) b_i$$

V. The security market line says that, in equilibrium, the return on security *i* is equal to the risk-free rate (R_f) plus the excess return on the market portfolio times the beta of security *i.*