EXAMPLE 7.1

RISK AND RETURN FOR A PORTFOLIO OF TWO ASSETS

Consider two assets, labeled 1 and 2, such that the return on asset 1 can take on $n = 3$ values and the return on asset 2 can take on $m = 3$ values. The $p(R_{1,i}, R_{2,j})$ probabilities in the joint probability distribution of the returns are given by the entries in the following table:

i	$R_{1,i}$ \ j, $R_{2,j}$	1: .0	2: .20	3: .40
1	.0	.05	.05	.00
2	.10	.05	.30	.15
3	.20	.05	.15	.20

Thus, for example, the probability that $R_1 = .10$ and $R_2 = .20$ is .30, or 30 percent. This figure is located where $i = 2$ and $j = 2$ in the table.

From Equation (7.4a) we have as the expected return on the first asset

$$\begin{aligned}\mu_1 &= R_{1,1} \sum_{j=1}^{m} p(R_{1,1}, R_{2,j}) \\ &\quad + R_{1,2} \sum_{j=1}^{m} p(R_{1,2}, R_{2,j}) \\ &\quad + R_{1,3} \sum_{j=1}^{m} p(R_{1,3}, R_{2,j}) \\ &= .00(.05 + .05 + .00) \\ &\quad + .10(.05 + .30 + .15) \\ &\quad + .20(.05 + .15 + .20) \\ &= .13\end{aligned}$$

and similarly, $\mu_2 = .24$.

From Equation (7.7a) we have as the variance of return on the first asset

$$\begin{aligned}\sigma_1^2 &= (R_{1,1} - \mu_1)^2 \sum_{j=1}^{m} p(R_{1,1}, R_{2,j}) \\ &\quad + (R_{1,2} - \mu_1)^2 \sum_{j=1}^{m} p(R_{1,2}, R_{2,j}) \\ &\quad + (R_{1,3} - \mu_1)^2 \sum_{j=1}^{m} p(R_{1,3}, R_{2,j}) \\ &= (-.13)^2(.05 + .05 + .00) \\ &\quad + (-.03)^2(.05 + .30 + .15) \\ &\quad + (.07)^2(.05 + .15 + .20) \\ &= .0041\end{aligned}$$

so that $\sigma_1 = .064$. Similarly, $\sigma_2^2 = .0184$ and $\sigma_2 = .136$. Note that $\mu_1 < \mu_2$, so that asset 2 has a greater expected return, but $\sigma_1 < \sigma_2$, so that the return on asset 2 is relatively more uncertain.

EXAMPLE 7.1 *(Continued)*

From Equation (7.8) we have as the covariance of returns on the two assets

$$\begin{aligned}\text{Cov}\,[R_1, R_2] &= \sum_{i=1}^{n}\sum_{j=1}^{m} (R_{1,i} - \mu_1)(R_{2,j} - \mu_2)p(R_{1,i}, R_{2,j}) \\ &= -.13(-.24)(.05) + -.13(-.04)(.05) + -.13(.16)(.00) \\ &\quad + -.03(-.24)(.05) + -.03(-.04)(.30) + -.03(.16)(.15) \\ &\quad + .07(-.24)(.05) + .07(-.04)(.15) + .07(.16)(.20) \\ &= .0028\end{aligned}$$

For a portfolio composed of assets 1 and 2, the relation between portfolio composition and portfolio risk and return is described by the pair of equations

$$\begin{aligned}\mu &= x_1\mu_1 + x_2\mu_2 \\ &= x_1(.13) + x_2(.24) \\ \sigma^2 &= x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\,\text{Cov}\,[R_1, R_2] \\ &= x_1^2(.0041) + x_2^2(.0184) + 2x_1x_2(.0028)\end{aligned}$$

If we consider eleven different portfolio allocations, given by $x_2 = .0, .1, .2, \ldots, .9, 1.0$ and $x_1 = 1 - x_2$, we can compute the (μ, σ) pairs as follows:

x_1	x_2	μ	σ
1.0	.0	.130	.0640
.9	.1	.141	.0633
.8	.2	.152	.0652
.7	.3	.163	.0696
.6	.4	.174	.0759
.5	.5	.185	.0838
.4	.6	.196	.0929
.3	.7	.207	.1028
.2	.8	.218	.1133
.1	.9	.229	.1243
.0	1.0	.240	.1360

This table shows explicitly the trade-off between portfolio risk and portfolio return as the composition of the portfolio is varied.

sociated with each of the nm events. For the event where $R_1 = R_{1,i}$ and $R_2 = R_{2,j}$, the number is $R_{1,i}$, the realized return on the first asset. In computing μ_2, the number associated with the same event is $R_{2,j}$, the realized return on the second asset. Example 7.1 shows how Equation (7.4) is used to compute the expected return on an asset from a given probability distribution of pairs of returns.

We can also define the expected return on the portfolio as a whole. If the fractional portfolio allocation is (x_1, x_2), then the portfolio return which occurs from a realization of the event where $R_1 = R_{1,i}$ and $R_2 = R_{2,j}$ is $x_1R_{1,i} + x_2R_{2,j}$.