# Note on Covariance and Correlation

William L. Silber and Jessica Wachter

## 1. Theoretical Definitions

The covariance of two random variables, $R_1$ and $R_2$, is defined as:

$$Cov(R_1, R_2) = E[(R_1 - m_1)(R_2 - m_2)]$$

The covariance can be calculated as follows, where the $p_{ij}$ are the elements of a joint probability distribution:

$$Cov(R_1, R_2) = \sum_i \sum_j (R_{1i} - m_1)(R_{2j} - m_2) p_{ij}$$

In words, the covariance is calculated by summing the product of the paired deviations of each observation from its respective mean, with each pair multiplied by its probability.

Correlation, $r$, is defined as:

$$r = \frac{Cov(R_1, R_2)}{s_{R_1} \cdot s_{R_2}}$$

## 2. Numerical Examples

Suppose one of three things will happen next year. There will either be a recession, a boom, or things will continue as normal. Each of these scenarios (or observations) is equally likely. That is, each happens with probability 1/3 (these are the $p_{ij}$).

**(a)** First, let's compare a fund that invests in U.S. stocks to a fund that invests in U.S. bonds. Based on Table I, and using the 1/3 probability associated with each scenario, we calculate a mean return on the stock fund of .11.  The mean return on the bond fund is .07. Table I also records the deviations from the mean for each scenario.

Table I

| | U.S. Stock Fund | | U.S. Bond Fund | |
|---|---|---|---|---|
| Scenario | Return | Deviation from Mean | Return | Deviation from Mean |
| Recession | -.07 | -.18 | .17 | .10 |
| Normal | .12 | .01 | .07 | 0 |
| Boom | .28 | .17 | -.03 | -.10 |

Given the definition of covariance from above (the sum of paired deviations from the means multiplied by the respective probabilities), we have the covariance between the U.S. stock fund and the U.S. bond fund as:

$$\frac{1}{3}(-.18)(.10)+\frac{1}{3}(.01)(0)+\frac{1}{3}(.17)(-.10)=-.01167$$

The definition of correlation from above is:

$$r=\frac{Cov}{s_{stock}\cdot s_{bond}}$$

In this case, $s_{stock}=.14306$, $s_{bond}=.08165$, hence the correlation is:

$$r=\frac{-.01167}{(.14306)(.08165)}=-.999$$

**(b)** Now compare the U.S. stock fund with a fund designed to track small capitalization stocks.  Based on Table II we calculate that the mean return on this “small cap” fund is .15.

Table II

| Scenario | U.S. Stock Fund Return | U.S. Stock Fund Deviation from Mean | Small Cap Fund Return | Small Cap Fund Deviation from Mean |
|---|---|---|---|---|
| Recession | -.07 | -.18 | -.15 | -.30 |
| Normal | .12 | .01 | .19 | .04 |
| Boom | .28 | .17 | .41 | .26 |

The covariance between the U.S. stock fund and the small cap fund is:

$$\frac{1}{3}(-.18)(-.30)+\frac{1}{3}(.01)(.04)+\frac{1}{3}(.17)(.26)=.0329$$

The correlation is:

$$\frac{Cov}{s_{stock}\cdot s_{small\ cap}}=.997$$

**(c)** Finally, compare the U.S. stock fund with a fund that invests in Russian bonds. Based on Table III we calculate that the mean return on the Russian bond fund is .02.

Table III

| Scenario | U.S. Stock Fund Return | U.S. Stock Fund Deviation from Mean | Russian Bond Fund Return | Russian Bond Fund Deviation from Mean |
|---|---|---|---|---|
| Recession | -.07 | -.18 | .27 | .25 |
| Normal | .12 | .01 | -.48 | -.50 |
| Boom | .28 | .17 | .27 | .25 |

The covariance between the U.S. stock fund and the Russian bond fund is:

$$\frac{1}{3}(-.18)(.25)+\frac{1}{3}(.01)(-.50)+\frac{1}{3}(.17)(.25)=-.0025$$

The correlation is:

$$\frac{Cov}{s_{stock}\cdot s_{Russian\ bond}}=-.0494$$

**3. General Rules**

Negative Correlation:

In case 2(a) we see that the stock and bond funds move in opposite directions relative to their means across the different scenarios. In particular, according to Table I, during recession stocks earn 18% below their average return (-.07 versus an average of .11) while the bond fund earns 10% above its average (.17 versus and average of .07). The opposite occurs in the case of a boom. Thus, the covariance and correlation are negative because:

| | | | |
|---|---|---|---|
| When | $R_S > \mu_S$ | we have | $R_B < \mu_B$ |
| When | $R_S < \mu_S$ | we have | $R_B > \mu_B$ |

Positive Correlation

In case 2(b) we see that the stock fund and small capitalization stocks move in the same direction relative to their means across the different scenarios (you fill in the details). We have positive covariance and correlation because:

| | | | |
|---|---|---|---|
| When | $R_S > \mu_S$ | we have | $R_{SC} > \mu_{SC}$ |
| When | $R_S < \mu_S$ | we have | $R_{SC} < \mu_{SC}$ |

Zero (Very Low) Correlation

In case 2(c) we see that the stock fund and the Russian bond fund do not move in a reliable relationship relative to their means across the different scenarios. In particular, in recession the stock fund is below its mean and the Russian fund is above, while in boom times both are above their means. Thus, the covariance is small (because the components of the calculation offset one another) and the correlation is close to zero. The same would be true if the relationship were reversed. Hence, covariance is small and correlation is close to zero if:

| | | | |
|---|---|---|---|
| When | $R_S > \mu_S$ | we have | $R_{RB} \gtrless \mu_{RB}$ |
| When | $R_S < \mu_S$ | we have | $R_{RB} \gtrless \mu_{RB}$ |