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Foundations of Finance (**COR1-GB.2311**)

Equilibrium Term Structure under the Expectations Theory

PART I

Given: ${}_tR_1 = .02$; and Expected ${}_{t+1}R_1 = .04$

To Prove: Equilibrium ${}_tR_2 = [(1 .02)(1.04)]^{1/2} - 1 = .02995 \approx .03$

Approach: Assume ${}_tR_2 = .02995 = .03$ and examine investor holding period yields (HPY). If short-term (one-year) investors realize the same HPY by buying ${}_tR_1$ compared with buying ${}_tR_2$ and selling after one year, they will be indifferent between investing in ${}_tR_1$ and investing in ${}_tR_2$. And if long-term (two-year) investors realize the same HPY by buying ${}_tR_2$ compared with buying ${}_tR_1$ and reinvesting at Exp ${}_{t+1}R_1$ they too will be indifferent between investing in ${}_tR_1$ compared with ${}_tR_2$. This would imply no buying or selling pressure in the marketplace, hence prices and yields would be equilibrium.

Proof: Uses Pure Discount Bonds

NB: ${}_tR_1 = .02$ implies $P_1 = \dfrac{100}{1.02} = 98.0392$

${}_tR_2 = .03$ implies $P_2 = \dfrac{100}{(1.03)^2} = 94.2596$

ONE-YEAR INVESTOR ALTERNATIVE STRATEGIES

Strategy 1: Buy ${}_tR_1$ at 98.0392 and hold to maturity

$$HPY = \frac{100}{98.0392} - 1 = .02$$

Strategy 2: Buy ${}_tR_2$ at 94.2596 and sell after one year. Because ${}_tR_2$ becomes a one-year bond after one year and because one-year bonds next year are expected to yield 4%, the expected selling price of ${}_tR_2$ next year is:

$$\frac{100}{1.04} = 96.154$$

$$HPY = \frac{96.154}{94.2596} - 1 = .02$$

TWO-YEAR INVESTOR ALTERNATIVE STRATEGIES

Strategy 1: Buy ${}_tR_1$ and reinvest at Exp ${}_{t+1}R_1$.

$$HPY = \left[(1.02)(1.04)\right]^{1/2} - 1 = .02995 \approx .03$$

Strategy 2: Buy ${}_tR_2$ and hold to maturity

$$HPY = \left(\frac{100}{94.2596}\right)^{1/2} - 1 = .02999 \approx .03$$

Conclusion

Because HPY's for all investors and all strategies are equal, the term structure is in equilibrium with ${}_tR_2$ a geometric average of ${}_tR_1$ and Exp ${}_{t+1}R_1$.

PART II

Question

What happens to equilibrium ${}_tR_2$ when expectations change? In particular, when Exp ${}_{t+1}r_1$ = .06, what is the new equilibrium ${}_tR_2$? How does it come about?

Given: ${}_tR_1$ = .02 and the new Exp ${}_{t+1}R_1$ = .06

To Prove: The new equilibrium ${}_tR_2 = [(1.02)(1.06)]^{1/2} - 1 = .0398$ because portfolio adjustments by market participants will make it so.

Approach: Assume for a moment that P_2 (the price of ${}_tR_2$) remains at 94.2596 so that ${}_tR_2$ remains at .03. Examine what portfolio adjustments "two-year" investors will undertake and see what impact that will have on P_2. After determining the new equilibrium P_2 that leaves two-year investors indifferent between both of their investment strategies, see if that P_2 "works" for one-year investors. If so, that is the new equilibrium P_2 with the associated new equilibrium ${}_tR_2$.

Proof:

TWO-YEAR INVESTOR STRATEGIES

1. Two-year investors prefer ${}_tR_1$ and reinvesting in new Exp ${}_{t+1}R_1$

HPY = $[(1.02)(1.06)]^{1/2}$ -1 = .0398

To buying ${}_tR_2$ at the old P_2 = 94.2596

$$HPY = \left(\frac{100}{94.2596}\right)^{1/2} - 1 = .092999 \approx .03$$

2. Therefore, two-year investors want to sell ${}_tR_2$ at P_2 = 94.2596. This selling pressure drives down P_2 until two-year investors are willing to hold ${}_tR_2$. That occurs when its new HPY = .0398.

Therefore, the new equilibrium P_2 is

$$P_2 = \frac{100}{(1.0398)^2} = 92.49$$

At this new price the yield on ${}_tR_2$ is:

$$HPY = \left(\frac{100}{92.,49}\right)^{1/2} - 1 = .0398$$

Thus two-year investors are indifferent between ${}_tR_1$ and ${}_tR_2$ when P_2 =92.49.

ONE-YEAR INVESTOR STRATEGIES

With P_2 = 92.49 and ${}_tR_2$ = .0398 one-year investors are also indifferent between both of their strategies, as shown in the following 2 possibilities:

1. Buy ${}_tR_1$ and earn .02

2. Buy ${}_tR_2$ and sell after one year. The expected selling price of ${}_tR_2$ after one year is now:

$$\frac{100}{1.06} = 94.3396$$

Therefore

$$HPY = \frac{94.3396}{92.49} - 1 = .01999 \approx .02$$

Therefore,

New equilibrium $P_2 = 92.49$

New equilibrium ${}_tR_2 = .0398$

Conclusion

Market forces in the form of portfolio adjustments by investors drive long-term rates into an average of current and expected future short-term rates.